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Memoir on the Theory of the Partitions of Numbers. Part V: Partitions in Two-Dimensional Space

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III. *Memoir on the Theory of the Partitions of Numbers.—Part V. Partitions in Two-dimensional Space.*

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Introduction.

In previous papers* I have broached the question of the two-dimensional partitions of numbers—or, say, the partitions in a plane—without, however, having succeeded in establishing certain conjectured formulas of enumeration. The parts of such partitions are placed at the nodes of a complete, or of an incomplete, lattice in two dimensions, in such wise that descending order of magnitude is in evidence in each horizontal row of nodes and in each vertical column. No decided advance was made in regard to the complete lattice, and the question of the incomplete lattice is considered for the first time in the present paper.

I return to the subject because I am now able to throw a considerable amount of fresh light upon the problem, and have succeeded in overcoming most of the difficulties which surround it. In fact, I am now able to show how the generating functions may be constructed in respect of any lattice, complete or incomplete, in forms which are free from redundant terms. I have not succeeded, so far, in giving a general algebraic expression to the functions, but, in the case of the complete lattice, I have shown that an assumption as to form, consistent with all results that have been arrived at in particular cases, leads at once to the expression that has been for so long the conjectured result. For the complete lattice of two rows, and for the incomplete lattice of two rows, the results have been obtained without any assumption in regard to form, and must be regarded as rigidly established.

Before proceeding to explain the new method of research which enables this paper to make a notable advance, I must hasten to correct an error which I had not detected at the time a former paper was written.

It will be remembered that partitions in a plane are such that there is a graphical

* "Memoir on the Theory of the Partitions of Numbers," 'Phil. Trans. Roy. Soc.,' A, 1896, vol. 187, pp. 619–673; 1899, vol. 192, pp. 351–401; 1905, vol. 205, pp. 37–59.

representation by nodes upon a three-dimensional lattice, just as for partitions on a line there is a graphical representation by nodes upon a two-dimensional lattice. It is convenient to replace these nodes by units, and to regard partitions on a line as being in one-to-one correspondence with partitions in a plane when the part magnitude of such is restricted to be not greater than unity; thus, instead of saying with FERRERS that

. . .
. .
.

is a graphical representation of the line partition 321, I regard the plane partition of units

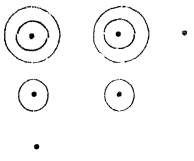
111
11
1

as being in one-to-one correspondence with the line partition.

Just so the plane partition

331
22
1

is graphically represented by piles of nodes perpendicular to the plane of the paper, say



or we may replace the nodes by units, and say that it is in one-to-one correspondence with a space partition, the part magnitude being restricted to unity. The plane partition arises by projection of the space partition upon one of the co-ordinate planes, just as the line partition arises by projection of the plane partition, with which it is in correspondence, upon one of the co-ordinate axes.

Every two-dimensional graph of nodes may be interpreted either by rows or by columns, and every plane partition of units may be projected in two ways. The graphs *in solido* admit of one, *two*, three, or six readings.

In previous papers I omitted to notice that a three-dimensional graph may admit of two readings. The omission came to my notice when I was trying to verify that the number of partitions of w *in plano* the numbers of rows and columns, and also the part magnitude, being unrestricted is given by the coefficient of x^w in the ascending expansion of the algebraic fraction

$$\frac{1}{(1-x)\,(1-x^2)^2\,(1-x^3)^3\,(1-x^4)^4 \ldots \textit{ad inf.}}.$$

I counted, as far as weight 16, the numbers of the partitions by separately counting those whose graphs possess one, three, and six readings. At weight 13 a discrepancy appeared, because of that weight there are only two graphs which have one reading, and, on the assumption that the remaining graphs could be read in either three or six ways, it was clear that the number of the partitions must be $\equiv 2 \bmod 3$; but the coefficients of x^{13} in the *supposed* generating function was found to be 2485, which is $\equiv 1 \bmod 3$. It thus became clear either that the reasoning from the graphs was wrong, or that the generating function was at fault. The discrepancy was cleared up by the discovery that at weight 13 graphs with two readings present themselves for the first time. The simplest of these is

$$\begin{matrix} 331 \\ 211 \\ 2 \end{matrix} \quad \text{of weight 13.}$$

The property possessed by these partitions is that the successive rows are the conjugates of the successive columns without being identical with them; that is to say, that the successive rows are not to be self-conjugate partitions. Thus, 331, 211, 2 are conjugates of 322, 31, 11 respectively. The reading of the corresponding hree-dimensional graph in the six modes gives either

$$\begin{matrix} 331 \\ 211 \\ 2 \end{matrix} \quad \text{or} \quad \begin{matrix} 322 \\ 31 \\ 11. \end{matrix}$$

The separate enumeration of these forms is a matter for future enquiry.

Art. 1. Turning now to the substance of this communication, I shall introduce a new plan of procedure which is applicable when the places for the parts of the partitions are given by the nodes of two-dimensional lattices, which may be complete or incomplete. In every case I suppose the part magnitude to be not greater than l, and when the lattice is complete, I suppose it to have m rows and n columns. The generating function which gives by the coefficients of x^w the number of the partitions of w of the nature considered will be denoted by GF (l, m, n).

In the excellent notation of CAYLEY and SYLVESTER I shall denote the algebraic expression $1-x^s$ by $(\mathbf{s})$, employing Clarendon type for the letter s, and thus $1-x$ by $(\mathbf{1})$ and $1-x^{l+1}$ by $(\mathbf{l+1})$, using always the Clarendon type in order to differentiate such notation from that in which between the brackets the ordinary Roman type is employed; the latter will, in general, denote integers s, 1, $l+1$, as the case may be. The notation is perfect for the purpose in hand, because it merely exhibits and concentrates attention upon the exponent s, which is the essential part of the expression, and the only part that in many cases it is necessary to handle algebraically. Further, in several instances, identities involving such expressions in

Clarendon type can be transformed into identities involving Roman type by simply changing the type in the bracketed factors; this ensues because the fractions $(\mathbf{s})\div(\mathbf{t})$ in Clarendon becomes equal to $(s)\div(t)$ in Roman in the limit when x is equal to unity.

Art. 2. The graphical representation by a three-dimensional lattice shows that the generating function GF (l, m, n) is unaltered by any permutation of the letters l, m, n. The subjoined notation is designed to show clearly the six alternative expressions of the generating function, arising from this circumstance, which it is a principal object of this paper to establish.

Thus, write

$$|\mathrm{LM}|_s = \frac{(\mathbf{l+s})\,(\mathbf{l+s+1})\ldots(\mathbf{l+m+s-1})}{(\mathbf{s})\,(\mathbf{s+1})\ldots(\mathbf{m+s-1})},$$

$$|\mathrm{ML}|_s = \frac{(\mathbf{m+s})\,(\mathbf{m+s+1})\ldots(\mathbf{m+l+s-1})}{(\mathbf{s})\,(\mathbf{s+1})\ldots(\mathbf{l+s-1})},$$

$$|\mathrm{MN}|_s = \frac{(\mathbf{m+s})\,(\mathbf{m+s+1})\ldots(\mathbf{m+n+s-1})}{(\mathbf{s})\,(\mathbf{s+1})\ldots(\mathbf{n+s-1})},$$

$$|\mathrm{NM}|_s = \frac{(\mathbf{n+s})\,(\mathbf{n+s+1})\ldots(\mathbf{n+m+s-1})}{(\mathbf{s})\,(\mathbf{s+1})\ldots(\mathbf{m+s-1})},$$

$$|\mathrm{NL}|_s = \frac{(\mathbf{n+s})\,(\mathbf{n+s+1})\ldots(\mathbf{n+l+s-1})}{(\mathbf{s})\,(\mathbf{s+1})\ldots(\mathbf{l+s-1})},$$

$$|\mathrm{LN}|_s = \frac{(\mathbf{l+s})\,(\mathbf{l+s+1})\ldots(\mathbf{l+n+s-1})}{(\mathbf{s})\,(\mathbf{s+1})\ldots(\mathbf{n+s-1})}.$$

It is to be shown that

$$\begin{aligned}
\mathrm{GF}(l, m, n) &= |\mathrm{LM}|_1\ |\mathrm{LM}|_2\ \ldots\ |\mathrm{LM}|_n, \\
&= |\mathrm{ML}|_1\ |\mathrm{ML}|_2\ \ldots\ |\mathrm{ML}|_n, \\
&= |\mathrm{MN}|_1\ |\mathrm{MN}|_2\ \ldots\ |\mathrm{MN}|_l, \\
&= |\mathrm{NM}|_1\ |\mathrm{NM}|_2\ \ldots\ |\mathrm{NM}|_l, \\
&= |\mathrm{NL}|_1\ |\mathrm{NL}|_2\ \ldots\ |\mathrm{NL}|_m, \\
&= |\mathrm{LN}|_1\ |\mathrm{LN}|_2\ \ldots\ |\mathrm{LN}|_m.
\end{aligned}$$

Art. 3. Every known particular case agrees with these formulæ, but only two general results have been established prior to this paper. One is the well-known case of partitions on a line, viz. :—

$$\mathrm{GF}(l, 1, n) = \frac{(\mathbf{l+1})\,(\mathbf{l+2})\ldots(\mathbf{l+n})}{(\mathbf{1})\,(\mathbf{2})\ldots(\mathbf{n})} = \frac{(\mathbf{n+1})\,(\mathbf{n+2})\ldots(\mathbf{n+l})}{(\mathbf{1})\,(\mathbf{2})\ldots(\mathbf{l})},$$

$$= |\mathrm{LN}|_1 = |\mathrm{NL}|_1,$$

and the other is that given in Part II. of this Memoir, and also by FORSYTH,

$$\mathrm{GF}\,(\infty, 2, n) = \frac{1}{(1)\,\{(2)\,(3)\,\ldots\,(n)\}^2\,(n+1)} = |\,\mathrm{LN}\,|_1\,|\,\mathrm{LN}\,|_2, \text{ when } l = \infty.$$

This generating function may be regarded as enumerating partitions—

(i) At the nodes of a lattice of 2 rows and n columns (or of n rows and 2 columns)

. . . . .
. . . . .

the part magnitude being unrestricted;

(ii) At the nodes of a lattice which has the number of $\frac{\text{rows}}{\text{columns}}$ unrestricted, the number of $\frac{\text{columns}}{\text{rows}}$ equal to n and the part magnitude restricted to be $\not> 2$.

The found result shows that the number of partitions of w is equal to the number of ways of composing w with

one kind of unity,
two kinds of twos,
two kinds of threes,
. . . . . .
two kinds of n's,
one kind of $n+1$,

but all attempts to establish a one-to-one correspondence have failed. Had this proved to have been feasible it *might* have been extended to prove the similar results for GF (∞, m, n) where $m > 2$.

Art. 4. The linear Diophantine Analysis, which was applied to the same question in an earlier part of this Memoir, having also failed to establish general results, recourse has been had to a plan suggested by Part IV. of the Memoir,* and a considerable advance has been made. In that paper I considered the number of different ways in which k *different* numbers can be placed at the nodes of a lattice, complete or incomplete, the number of nodes being k, and the numbers being placed in such wise that descending order of magnitude is in evidence in each row from West to East and in each column from North to South.

In the paper quoted I showed that if the rows involve $a_1, a_2, \ldots, a_m$ nodes respectively, where, of course, $a_1 \geq a_2 \geq \ldots \geq a_m$, the number of ways of arranging the Σa different numbers at the nodes is

$$\frac{(\Sigma a)\,!}{(a_1+m-1)!\;(a_2+m-2)!\;\ldots\;(a_{m-1}+1)!\;a_m!}\;\Pi_{s,}\,(a_s - a_t - s + t),$$

* "Memoir on the Theory of the Partitions of Numbers," 'Phil. Trans.,' A, 1908, vol. 209, pp. 153–175.

where $s < t$ and the product Π has reference to every pair of numbers a_s, a_t drawn from the succession $a_1, a_2, \ldots, a_m$.

This result will be found to furnish an important key to the solution of the problems before us.

It is possible, by the method employed, to consider the generating functions for partitions at the nodes of an incomplete lattice, and I shall use GF $(l; a, b, c, \ldots)$ to denote that which has reference to a lattice whose successive rows involve $a, b, c, \ldots$ nodes, respectively, the part magnitude being restricted by the number l. In this notation GF (l, m, n) may alternately be written GF $(l; n^m)$ or GF $(l; m^n)$, wherein n^m will denote m rows each of n nodes.

I derive from every lattice, complete or incomplete, a lattice-function of x, and this function depends, like the generating function, not only upon the specification of the lattice, but also upon the number l which limits the part magnitude. I denote this function by L (l, m, n) or by L $(l; a, b, c, \ldots)$, according as the lattice is complete or incomplete. In cases where no confusion can arise, I simply write L for brevity.

Art. 5. I will now explain the formation of the functions

$$\mathrm{L}(\infty, m, n) \quad \text{and} \quad \mathrm{L}(\infty; a, b, c, \ldots);$$

and then establish the fundamental propositions

$$\mathrm{GF}(\infty, m, n) = \frac{\mathrm{L}(\infty, m, n)}{(1)(2)\ldots(mn)},$$

$$\mathrm{GF}(\infty; a, b, c, \ldots) = \frac{\mathrm{L}(\infty; a, b, c, \ldots)}{(1)(2)\ldots(\Sigma a)}.$$

In the next place I will explain the formation of the functions

$$\mathrm{L}(l, m, n) \quad \text{and} \quad \mathrm{L}(l; a, b, c, \ldots),$$

and establish the fundamental propositions

$$\mathrm{GF}(l, m, n) = \frac{\mathrm{L}(l, m, n)}{(1)(2)\ldots(mn)},$$

$$\mathrm{GF}(l; a, b, c, \ldots) = \frac{\mathrm{L}(l; a, b, c, \ldots)}{(1)(2)\ldots(\Sigma a)}.$$

Art. 6. Consider an incomplete lattice having 3, 2, 1 nodes in the rows respectively, and let any six different integers (say the first six) be placed in any manner at the nodes in such wise that descending order of magnitude is in evidence in each row and in each column; such an arrangement may be

$$\begin{matrix} 6\,3\,1 \\ 5\,2 \\ 4 \end{matrix}$$

Let the Greek letters α, β, γ be associated with the first, second, and third rows, respectively, and consider each number in the lattice in succession in descending order of magnitude. Thus, beginning with 6: since it is in the first row I commence a succession of Greek letters with α; passing to 5, since it is in the second row, I follow with β; then 4 gives γ, since it is in the third row; then 3 gives α; 2, β; and finally 1 gives α.

$$\alpha\beta\gamma\alpha\beta\alpha.$$

In this way I obtain a permutation of the letters in $\alpha^3\beta^2\gamma$, where the exponents 3, 2, 1 enumerate the nodes in the successive rows of the lattice. This permutation possesses the property:—

> "If a dividing line be made between any two adjacent letters of the permutation, the succession of letters to the left of the dividing line is like the whole permutation, such that α occurs at least as often as β, β at least as often as γ; in other words, the numbers which specify the occurrences of α, β, γ, are in descending order of magnitude."

In fact, if the process of forming the Greek letter succession (or permutation) be arrested at any point, the lattice numbers that have been dealt with occupy a set of nodes which also constitute a lattice, complete or incomplete.

It follows, of course, that the first letter of the permutation must be α. The lattice arrangement of numbers is recoverable from the permutation, for it is merely necessary to write the numbers in descending order underneath the letters when we see that the successive lattice rows are indicated by the letters α, β, γ, respectively,

$$\begin{matrix}\alpha\beta\gamma\alpha\beta\alpha\\ 6\,5\,4\,3\,2\,1\end{matrix}.$$

The process is thus unique, and there will be as many different Greek letter permutations having the properties above specified as of arrangements of unequal numbers at the nodes of the lattice having the specified descending orders.

Every Greek letter permutation can be separated into groups, each of which contains letters in alphabetical order; in the case before us this is accomplished by two dividing lines

$$\alpha\beta\gamma\,|\,\alpha\beta\,|\,\alpha,$$

each of which separates a letter from one which follows it, but is prior to it in alphabetical order.

I associate a power of x with each permutation by taking for the exponent a sum of numbers $p_1+p_2+p_3+\ldots$, where p_s denotes that the s^{th} dividing line has p_s letters to the left of it. Thus in the above instance $p_1 = 3$, $p_2 = 5$, and the associated power of x is $x^{3+5} = x^8$.

Every one of the

$$\frac{(\Sigma\alpha)!}{(\alpha_1+m-1)!\,(\alpha_2+m-2)!\,\ldots\,(\alpha_{m-1}+1)!\,\alpha_m!}\prod_{s,t}(\alpha_s-\alpha_t-s+t)$$

arrangements of the different integers at the nodes of the lattice will thus have a power of x associated with it, and taking the sum of them all I obtain the lattice function

$$\mathrm{L}(\infty\,;\,\alpha_1,\,\alpha_2,\,\alpha_3,\,\ldots)=\Sigma x^{p_1+p_2+p_3+\ldots}.$$

Art. 7. I will set out at length the formation of $\mathrm{L}(\infty\,;\,3,\,2,\,1)$.

654	654	653	653	652	652	651	651
32	31	42	41	43	41	43	42
1	2	1	2	1	3	2	3
$\alpha\alpha\alpha\beta\beta\gamma$,	$\alpha\alpha\alpha\beta\gamma\,\vert\,\beta$,	$\alpha\alpha\beta\,\vert\,\alpha\beta\gamma$,	$\alpha\alpha\beta\,\vert\,\alpha\gamma\,\vert\,\beta$,	$\alpha\alpha\beta\beta\,\vert\,\alpha\gamma$,	$\alpha\alpha\beta\gamma\,\vert\,\alpha\beta$,	$\alpha\alpha\beta\beta\gamma\,\vert\,\alpha$,	$\alpha\alpha\beta\gamma\,\vert\,\beta\,\vert\,\alpha$
x^0	x^5	x^3	x^8	x^4	x^4	x^5	x^9

631	632	641	642	641	643	642	643
52	51	52	51	53	51	53	52
4	4	3	3	2	2	1	1
$\alpha\beta\gamma\,\vert\,\alpha\beta\,\vert\,\alpha$,	$\alpha\beta\gamma\,\vert\,\alpha\alpha\beta$,	$\alpha\beta\,\vert\,\alpha\gamma\,\vert\,\beta$,	$\alpha\beta\,\vert\,\alpha\gamma\,\vert\,\alpha\beta$,	$\alpha\beta\gamma\,\vert\,\alpha$,	$\alpha\beta\,\vert\,\alpha\alpha\gamma\,\vert\,\beta$,	$\alpha\beta\,\vert\,\alpha\beta\,\vert\,\alpha\gamma$,	$\alpha\beta\,\vert\,\alpha\alpha\beta\gamma$
x^8	x^3	x^{11}	x^6	x^7	x^7	x^6	x^2

so that

$$\mathrm{L}(\infty\,;\,3,\,2,\,1)=1+x^2+2x^3+2x^4+2x^5+2x^6+2x^7+2x^8+x^9+x^{11}.$$

It is obvious that this process can be carried out in respect of any lattice, complete or incomplete, and that the number of different Greek letters involved will be equal to the number of rows.

Art. 8. To show the connexion between such a lattice function and the corresponding generating of partitions at the nodes of the lattice I proceed as follows:—

We have to establish the relation

$$\mathrm{GF}(\infty\,;\,\alpha_1,\,\alpha_2,\,\alpha_3,\,\ldots)=\frac{\mathrm{L}(\infty\,;\,\alpha_1,\,\alpha_2,\,\alpha_3,\,\ldots)}{(1)\,(2)\,\ldots\,(\Sigma\mathrm{a})}.$$

As the simplest possible case (with a trivial exception) consider the complete lattice of 2 rows and 2 columns

. .
. .

and any numbers, equal or unequal, to be placed at the four nodes in such wise that

there is descending order of magnitude in both rows and in both columns; say that the numbers are

$$\begin{matrix} p & q \\ r & s \end{matrix}$$

subject to the conditions $p \geqq q \geqq s$, $p \geqq r \geqq s$.

It is clear that we must either have

$$\text{(i) } p \geqq q \geqq r \geqq s \quad \text{or} \quad \text{(ii) } p \geqq r > q \geqq s;$$

and that these two systems do not overlap.

If (i) obtains we may perform the summation $\Sigma x^{p+q+r+s}$ by writing $r = s+\mathrm{A}$, $q = s+\mathrm{A}+\mathrm{B}$, $p = s+\mathrm{A}+\mathrm{B}+\mathrm{C}$, where A, B, C are arbitrary positive integers, zero included; the sum is thus

$$\Sigma x^{\mathrm{C}+2\mathrm{B}+3\mathrm{A}+4s};$$

and since C, B, A, s may each of them assume all values ranging from zero to infinity, the sum is clearly

$$\frac{1}{(1)(2)(3)(4)};$$

if, on the other hand, the parts of the partition have such values that (ii) obtains, we may write

$$q = s+\mathrm{A}, \quad r = s+\mathrm{A}+\mathrm{B}+1, \quad p = s+\mathrm{A}+\mathrm{B}+\mathrm{C}+1,$$

and we have the sum

$$\Sigma x^{\mathrm{C}+2\mathrm{B}+3\mathrm{A}+4s+2},$$

which is equal to

$$\frac{x^2}{(1)(2)(3)(4)}.$$

By addition we have

$$\mathrm{GF}(\infty; 2, 2) = \frac{1+x^2}{(1)(2)(3)(4)} = \frac{1}{(1)(2)^2(3)};$$

and it will be noted that $1+x^2 = \mathrm{L}(\infty; 2, 2)$, derived, as above, from the lattice arrangements

$$\begin{matrix} 43 & 42 \\ 21 & 31 \end{matrix}$$

$$\begin{matrix} \alpha\alpha\beta\beta & \alpha\beta\,|\,\alpha\beta \\ x^0 & x^2 \end{matrix}$$

The fact is that the alternatives (i) and (ii) exist *because* there are two lattice arrangements of *unequal* numbers, and the signs of equality and inequality are arranged in (i) and (ii) so that the required sum may be separated into two non-overlapping systems in correspondence with the lattice arrangements. The fact that

$r > q$ in (ii), q being a letter prior to r in alphabetical order, is in direct correspondence with the Greek letter permutation $\alpha\beta|\alpha\beta$, in which a β precedes an α which is prior to it in alphabetical order. Moreover, r occurring in the second place in the condition (ii), $p \geqq r > q \geqq s$ clearly contributes the integer 2 to the exponent $C+2B+3A+4s+2$. Thus the numerator finally determined is necessarily the lattice function $L(\infty ; 2, 2)$ found by the specified rules.

Art. 9. Next take a case which is not quite so simple

$$\begin{matrix} p & q & r \\ s & t & u, \end{matrix}$$

where $p \geqq q \geqq r$, $s \geqq t \geqq u$, $p \geqq s$, $q \geqq t$, $r \geqq u$; the associated lattice arrangements and the Greek letter permutations are

654	643	653	652	642
321	521	421	431	531
$\alpha\alpha\alpha\beta\beta\beta$	$\alpha\beta\|\alpha\alpha\beta\beta$	$\alpha\alpha\beta\|\alpha\beta\beta$	$\alpha\alpha\beta\beta\|\alpha\beta$	$\alpha\beta\|\alpha\beta\|\alpha\beta$
x^0	x^2	x^3	x^4	x^6

yielding

$$L(\infty ; 3, 3) = 1+x^2+x^3+x^4+x^6 = \frac{(5)(6)}{(2)(3)}.$$

We have five non-overlapping systems

(i) $p \geqq q \geqq r \geqq s \geqq t \geqq u$, $\quad \alpha\alpha\alpha\beta\beta\beta$,

(ii) $p \geqq s > q \geqq r \geqq t \geqq u$, $\quad \alpha\beta|\alpha\alpha\beta\beta$,

(iii) $p \geqq q \geqq s > r \geqq t \geqq u$, $\quad \alpha\alpha\beta|\alpha\beta\beta$,

(iv) $p \geqq q \geqq s \geqq t > r \geqq u$, $\quad \alpha\alpha\beta\beta|\alpha\beta$,

(v) $p \geqq s > q \geqq t > r \geqq u$, $\quad \alpha\beta|\alpha\beta|\alpha\beta$,

wherein the positions occupied by the symbol $>$ are to be compared with the positions of the dividing lines in the corresponding Greek letter permutations. It is clear that the summations derived from the systems (i), (ii), (iii), (iv), (v) give powers of x in the numerator of the generating function exactly corresponding to those which enter into the lattice function by the rules given. Hence

$$GF(\infty ; 3, 3) = \frac{L(\infty ; 3, 3)}{(1)(2)\ldots(6)} = \frac{(5)(6)}{(2)(3)} \cdot \frac{1}{(1)(2)\ldots(6)},$$

$$= \frac{1}{(1)(2)^2(3)^2(4)}.$$

This short demonstration suffices to establish the general relations

$$\mathrm{GF}(\infty\,;\,a_1, a_2, a_3, \ldots) = \frac{\mathrm{L}(\infty\,;\,a_1, a_2, a_3, \ldots)}{(1)(2)\ldots(\Sigma a)}.$$

$$\mathrm{GF}(\infty\,;\,m, n) = \frac{\mathrm{L}(\infty\,;\,m, n)}{(1)(2)\ldots(mn)}.$$

Art. 10. Remarkable properties of the lattice functions will present themselves as the investigation proceeds. A few observations may be usefully made at this point. In every case the zero power of x presents itself in correspondence with that permutation of the Greek letters which is in alphabetical order.

In the case of partitions on a line the lattice is a single row of nodes; the Greek letter succession is composed entirely of the letter α and the lattice function is unity.

A most useful property arises simply from the definition of the function, viz., putting x equal to unity we find that the sum of the coefficient is

$$\frac{(\Sigma\alpha)!}{(\alpha_1+m-1)!\,(\alpha_2+m-2)!\ldots(\alpha_{m-1}+1)!\,\alpha_m!}\prod_{s,t}(\alpha_s-\alpha_t-s+t),$$

a verification of constant service.

I seek a representation of the lattice function that shall be a constant reminder of this enumerating function, and with this object in view I write the latter in the form

$$\frac{(1\,.\,2\,.\ldots\,\Sigma\alpha)}{(m\,.\,m+1\,.\ldots\,\alpha_1+m-1)\,(m-1\,.\,m\,.\ldots\,\alpha_2+m-2)\ldots\ldots\{2\,.\,3\ldots(\alpha_{m-1}+1)\}\,(1\,.\,2\,.\ldots\,\alpha_m)}\cdot\frac{\prod_{s,t}(\alpha_s-\alpha_t+t-s)}{\prod_{s,t}(t-s)},$$

and I then write

$$\mathrm{L}(\infty\,;\,a_1, a_2, a_3, \ldots, a_m)$$

$$= \frac{(1)(2)\ldots(\Sigma a)}{(m)(m+1)\ldots(a_1+m-1).(m-1)(m)\ldots(a_2+m-2)\ldots\ldots(2)(3)\ldots(a_{m-1}+1).(1)(2)\ldots(a_m)}\,\mathrm{IL}(\infty\,;\,a_1, a_2, \ldots, a_m)$$

where the algebraic fraction on the dexter, which I term the outer lattice function, is of *fixed form*, and the remaining algebraic factor $\mathrm{IL}(\infty\,;\,a_1, a_2, \ldots, a_m)$, which I term the inner lattice function, has to be determined.

The outer function reduces to the corresponding part of the arithmetical function when x is put equal to unity; under the same circumstances the inner function reduces to the sum of its own coefficients, viz., to

$$\prod_{s,t}(\alpha_s-\alpha_t+t-s) \div \prod_{s,t}(t-s).$$

There is a convenience in thus postulating the expression of an outer lattice function, because in every known result in regard to complete lattices the inner

function turns out to be simply unity; a principal object of this investigation is to establish that for the complete lattice the inner function is invariably unity. This is consistent with the result conjectured in Art. 2.

In regard to incomplete lattices the inner function is unity in special cases. The determination of its form for the general incomplete lattice is apparently a very difficult matter, which is reserved for future consideration. Its actual form for the lattice of two unequal rows will be determined presently.

Art. 11. There is also a vitally essential representation of the lattice function as a sum of sub-lattice functions, which forms a natural bridge from the function GF $(\infty ; a_1, a_2, a_3, \ldots)$ to the general function GF $(l ; a_1, a_2, a_3, \ldots)$. When the lattice function was formed from the permutations of the Greek letters, every permutation had s dividing lines where s ranged from zero up to a maximum value μ, which has not yet been determined. That portion of the lattice function which is derived from those permutations which involve precisely s dividing lines I name the sub-lattice function of order s and write it

$$\mathrm{L}_s (\infty ; a_1 a_2 a_3 \ldots), \text{ or } \mathrm{L}_s (\infty ; m, n), \text{ or simply } \mathrm{L}_s,$$

if no confusion arises from the abbreviation.

Thus

$$\mathrm{L} = \sum_0^{\mu} \mathrm{L}_s.$$

In the elementary examples already dealt with

$$\begin{aligned} \mathrm{L}(\infty, 2, 2) &= \mathrm{L}_0(\infty ; 2, 2) + \mathrm{L}_1(\infty ; 2, 2), \\ &= 1 + x^2, \end{aligned}$$

$$\begin{aligned} \mathrm{L}(\infty, 2, 3) &= \mathrm{L}_0(\infty ; 2, 3) + \mathrm{L}_1(\infty ; 2, 3) + \mathrm{L}_2(\infty ; 2, 3), \\ &= 1 + x^2 + x^3 + x^4 + x^6, \end{aligned}$$

$$\begin{aligned} \mathrm{L}(\infty ; 3, 2, 1) &= \mathrm{L}_0 + \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3, \\ &= 1 + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + x^9 + x^{11}. \end{aligned}$$

It will be observed that L_0 is invariably unity.

Art. 12. In terms of these sub-lattice functions I now define the new and more general lattice function $\mathrm{L}(l ; a_1, a_2, a_3, \ldots)$, in which l replaces ∞. I write

$$\begin{aligned} \mathrm{L}(l ; a_1, a_2, a_3, \ldots) = {} & (l+1)(l+2)\ldots(l+\Sigma \mathfrak{a})\, \mathrm{L}_0(\infty ; a_1, a_2, a_3, \ldots) \\ & + (l)(l+1)\ldots(l+\Sigma \mathfrak{a}-1)\, \mathrm{L}_1(\infty ; a_1, a_2, a_3, \ldots) \\ & + \ldots \\ & + (l-\mu+1)(l-\mu+2)\ldots(l+\Sigma \mathfrak{a}-\mu)\, \mathrm{L}_\mu(\infty ; a_1, a_2, a_3, \ldots); \end{aligned}$$

and also a general sub-lattice function

$$\mathrm{L}_s(l ; a_1, a_2, a_3, \ldots) = (l-s+1)(l-s+2)\ldots(l+\Sigma \mathfrak{a}-s)\, \mathrm{L}_s(\infty ; a_1, a_2, a_3, \ldots).$$

Art. 13. The next step is to establish the fundamental relations

$$\mathrm{GF}(l;\, a_1, a_2, a_3, \ldots) = \frac{\mathrm{L}(l;\, a_1, a_2, a_3, \ldots)}{(1)(2)\ldots(\Sigma \mathbf{a})};$$

$$\mathrm{GF}(l, m, n) = \frac{\mathrm{L}(l, m, n)}{(1)(2)\ldots(\mathbf{mn})}.$$

We have to take account of the circumstance that the part magnitude is now restricted not to exceed l. Take again the case, previously considered, of two rows and three columns. I recall the five distinct parts of the summation

$$\begin{array}{lll}
\text{(i)} & p \geq q \geq r \geq s \geq t \geq u & \text{giving} \quad \Sigma x^{\mathrm{E}+2\mathrm{D}+3\mathrm{C}+4\mathrm{B}+5\mathrm{A}+6u}, \\
\text{(ii)} & p \geq s > q \geq r \geq t \geq u & ,, \quad \Sigma x^{\mathrm{E}+2\mathrm{D}+3\mathrm{C}+4\mathrm{B}+5\mathrm{A}+6u+2}, \\
\text{(iii)} & p \geq q \geq s > r \geq t \geq u & ,, \quad \Sigma x^{\mathrm{E}+2\mathrm{D}+3\mathrm{C}+4\mathrm{B}+5\mathrm{A}+6u+3}, \\
\text{(iv)} & p \geq q \geq s \geq t > r \geq u & ,, \quad \Sigma x^{\mathrm{E}+2\mathrm{D}+3\mathrm{C}+4\mathrm{B}+5\mathrm{A}+6u+4}, \\
\text{(v)} & p \geq s > q \geq t > r \geq u & ,, \quad \Sigma x^{\mathrm{E}+2\mathrm{D}+3\mathrm{C}+4\mathrm{B}+5\mathrm{A}+6u+6}.
\end{array}$$

For the condition (i) we put

$$t = u+\mathrm{A}, \qquad r = u+\mathrm{A}+\mathrm{B}, \qquad q = u+\mathrm{A}+\mathrm{B}+\mathrm{C}, \qquad s = u+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D},$$

$$p = u+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E},$$

from which it is clear that

$$u+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}$$

cannot exceed l in magnitude; hence the sum

$$\Sigma x^{\mathrm{E}+2\mathrm{D}+3\mathrm{C}+4\mathrm{B}+5\mathrm{A}+6u}$$

is the generating function of partitions on a line into l parts not exceeding 6 in magnitude, and is therefore

$$\frac{(l+1)(l+2)\ldots(l+6)}{(1)(2)\ldots(6)}.$$

Similarly in each of the cases (ii), (iii), (iv), belonging to $\mathrm{L}_1(\infty, 3, 2)$, we put $p = u+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}+1$, and it is clear that $u+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}$ cannot exceed $l-1$ in magnitude; the corresponding portion of the generating function is therefore

$$\frac{(l)(l+1)\ldots(l+5)}{(1)(2)\ldots(6)}\,\mathrm{L}_1(\infty, 3, 2).$$

Finally, since in (v) we put

$$p = u+\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}+2,$$

we obtain a part of the generating function

$$\frac{(l-1)\,(l)\,\ldots\,(l+4)}{(1)\,(2)\,\ldots\,(6)}\,\mathrm{L}_2\,(\infty,\,3,\,2).$$

Thence

$$\mathrm{GF}\,(l,\,3,\,2) = \frac{(l+1)\,\ldots\,(l+6)\,\mathrm{L}_0\,(\infty,\,3,\,2) + (l)\,\ldots\,(l+5)\,\mathrm{L}_1\,(\infty,\,3,\,2) + (l-1)\,\ldots\,(l+4)\,\mathrm{L}_2\,(\infty,\,3,\,2)}{(1)\,(2)\,\ldots\,(6)},$$

$$= \frac{\mathrm{L}\,(l,\,3,\,2)}{(1)\,(2)\,\ldots\,(6)}.$$

In general, when the lattice has Σa nodes, we have a set of inequalities belonging to $\mathrm{L}_s\,(\infty\,;\,a_1,\,a_2,\,a_3,\,\ldots)$ which give rise to the generating function

$$\frac{(l-s+1)\,(l-s+2)\,\ldots\,(l+\Sigma a-s)}{(1)\,(2)\,\ldots\,(\Sigma a)}\,\mathrm{L}_s\,(\infty\,;\,a_1,\,a_2,\,a_3,\,\ldots);$$

and thus the above-given fundamental relations are established.

Art. 14. The generating functions for two-dimensional partitions $\mathrm{GF}\,(l,\,m,\,n)$ has been found in terms of lattice functions in the form

$$\frac{(l+1)\,\ldots\,(l+mn)\,\mathrm{L}_0 + (l)\,\ldots\,(l+mn-1)\,\mathrm{L}_1 + \ldots\ldots + (l-\mu+1)\,\ldots\,(l-\mu+mn)\,\mathrm{L}_\mu}{(1)\,\ldots\,(mn)}.$$

If we subtract these partitions from those enumerated by $\mathrm{GF}\,(\infty,\,m,\,n)$, we are left with those partitions which contain one part at least equal to or greater than $l+1$. I shall show how to determine directly the generating function for these in terms of lattice functions. To lead up to the proof, I will give an inductive proof of the theorem—

$$\mathrm{GF}\,(l,\,1,\,n) = \frac{(l+1)\,\ldots\,(l+n)}{(1)\,\ldots\,(n)}.$$

Taking the parts at n nodes in one row

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the partitions which have a highest part equal to $l+1$ will be obtained by placing the part $l+1$ to the left of each of the partitions enumerated by $\mathrm{GF}\,(l+1,\,1,\,n-1)$.

Hence the whole of the partitions which have one part at least equal or greater than $l+1$ are enumerated by

$$\sum_{l}^{\infty} x^{l+1}\,\mathrm{GF}\,(l+1,\,1,\,n-1)$$

and

$$\mathrm{GF}\,(l,\,1,\,n) = \mathrm{GF}\,(\infty,\,1,\,n) - \sum_{l}^{\infty} x^{l+1}\,\mathrm{GF}\,(l+1,\,1,\,n-1).$$

Assume the truth of the theorem in the case of

$$\mathrm{GF}(l+1, 1, n-1)$$

for all values of l; then

$$\mathrm{GF}(l, 1, n) = \frac{1}{(1)\dots(n)} - \sum_{l}^{\infty} x^{l+1} \frac{(l+2)\dots(l+n)}{(1)\dots(n-1)}.$$

Putting $x^l = \theta$,

$$x^{l+1}\frac{(l+2)\dots(l+n)}{(1)\dots(n-1)} = \frac{\theta x}{(1)\dots(n-1)}\left\{1 - \theta x^2\frac{(n-1)}{(1)} + \theta^2 x^5 \frac{(n-1)(n-2)}{(1)(2)} - \dots + (-)^{n-1}\theta^{n-1}x^{\frac{1}{2}(n-1)(n+2)}\right\};$$

and, therefore,

$$\sum_{l}^{\infty} x^{l+1}\frac{(l+2)\dots(l+n)}{(1)\dots(n-1)} = \frac{\theta x}{(1)^2(2)\dots(n-1)} - \frac{\theta^2 x^3 \frac{(n-1)}{(1)}}{(1)(2)^2(3)\dots(n-1)} + \frac{\theta^3 x^6 \frac{(n-1)(n-2)}{(1)(2)}}{(1)(2)(3)^2(4)\dots(n-1)} - \dots + (-)^{n-1}\theta^n \frac{x^{\frac{1}{2}n(n+1)}}{(1)\dots(n)};$$

and thence

$$\frac{1}{(1)\dots(n)} - \Sigma x^{l+1}\frac{(l+2)\dots(l+n)}{(1)\dots(n-1)} = \frac{1}{(1)\dots(n)}\left\{1 - \theta x\frac{(n)}{(1)} + \theta^2 x^3 \frac{(n)(n-1)}{(1)(2)} - \dots + (-)^n \theta^n x^{\frac{1}{2}n(n+1)}\right\},$$

$$= \frac{(l+1)\dots(l+n)}{(1)\dots(n)}.$$

Hence

$$\mathrm{GF}(l, 1, n) = \frac{(l+1)\dots(l+n)}{(1)\dots(n)}$$

by induction.

To generalize this method, I take a lattice which is complete but for the node at the left-hand top corner

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and first determine the generating function for partitions such that the descending order of part magnitude is in evidence in each row and in each column. I take the number of rows to be m, and the number of columns n. A slight consideration shows that if L_s be the sub-lattice function of order s for the complete lattice, that of the

deficient lattice now under consideration is $x^{-s}L_s$; hence, if the part magnitude be unrestricted, the generating function is

$$\frac{1+x^{-1}L_1+x^{-2}L_2+\ldots+x^{-\mu}L_\mu}{(1)\ldots(mn-1)};$$

and if the part magnitude be restricted not to exceed l,

$$\frac{(l+1)\ldots(l+mn-1)+x^{-1}(l)\ldots(l+mn-2)\,L_1+\ldots+x^{-\mu}(l-\mu+1)\ldots(l-\mu+mn-1)\,L_\mu}{(1)\ldots(mn-1)}.$$

A simple example, that may be at once verified, is found by taking $m=n=2$ and the defective lattice

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Here $L_1 = L_\mu = x^2$ and the generating functions are

$$\frac{1+x}{(1)(2)(3)} = \frac{1}{(1)^2(3)},$$

$$\frac{(l+1)(l+2)(l+3)+x(l)(l+1)(l+2)}{(1)(2)(3)};$$

putting $l=1$ we obtain $1+2x+x^2+x^3$, verified by

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. . . . 1 . 1 . 1 1
1 x x x^2 x^3.

Now consider the partitions at the nodes of the complete lattice such that one part at least is equal to $l+1$ and no part exceeds $l+1$. We obtain all such by placing the part $l+1$ at the node situated at the left-hand top corner and connecting with it all of the partitions at the nodes of the incomplete lattice, which are such that the part magnitude is restricted not to exceed $l+1$ in magnitude.

We thus derive a generating function

$$x^{l+1}\frac{(l+2)\ldots(l+mn)+x^{-1}(l+1)\ldots(l+mn-1)\,L_1+\ldots\ldots+x^{-\mu}(l-\mu+2)\ldots(l-\mu+mn)}{(1)\ldots(mn-1)},$$

and thence the generating function, which enumerates all partitions at the nodes of the complete lattice, which are such that each has one or more parts at least as great as $l+1$, is

$$\Sigma x^{l+1}\frac{(l+2)\ldots(l+mn)+x^{-1}(l+1)\ldots(l+mn-1)\,L_1+\ldots\ldots+x^{-\mu}(l-\mu+2)\ldots(l-\mu+mn)}{(1)\ldots(mn-1)},$$

and it is easy to verify that this expression added to the expression already found for GF (l, m, n) is, in fact, equal to

$$\frac{L_0+L_1+\ldots+L_\mu}{(1)\ldots(mn)},$$

that is, to GF (∞, m, n).

Art. 15. This main proposition involves the whole theory of the partitions at the nodes of an incomplete lattice; it gives the true generating function without redundant terms, and this only needs examination and, where possible, simplification. Such simplification is apparently always possible when the lattice is complete. Moreover, there is the task of exhibiting $L(\infty ; a_1, a_2, a_3, \ldots)$ as a product of outer and inner lattice functions and of finding the algebraic expression of $L_s(\infty ; a_1, a_2, a_3, \ldots)$. There is an important and quite general property of the lattice function which must now be explained. If a lattice be read by columns instead of by rows its specification changes from a partition to the conjugate partition, and it is a trivial remark that the generating function of partitions at the nodes is not altered. In fact, if the rows possess $a_1, a_2, \ldots, a_m$ nodes and the columns $b_1, b_2, \ldots, b_n$ nodes

$$\mathrm{GF}(l ; a_1, a_2, \ldots, a_m) = \mathrm{GF}(l ; b_1, b_2, \ldots, b_n).$$

Moreover, since the generating function is the quotient of the lattice function by an algebraic function which depends merely upon the number of nodes, it is clear that

$$L(\infty ; a_1, a_2, \ldots, a_m) = L(\infty ; b_1, b_2, \ldots, b_n),$$

$$L(l ; a_1, a_2, \ldots, a_m) = L(l ; b_1, b_2, \ldots, b_n),$$

$(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_n)$ being conjugate partitions.

From the last written relation we find

$$\begin{aligned}(l+1)\ldots(l+\Sigma a)+(l)\ldots(l+\Sigma a-1)\,L_1(\infty ; a_1, a_2, \ldots, a_m)\\ +(l-1)\ldots(l+\Sigma a-2)\,L_2(\infty ; a_1, a_2, \ldots, a_m)+\ldots\\ =(l+1)\ldots(l+\Sigma a)+(l)\ldots(l+\Sigma a-1)\,L_1(\infty ; b_1, b_2, \ldots, b_n)\\ +(l-1)\ldots(l+\Sigma a-2)\,L_2(\infty ; b_1, b_2, \ldots, b_n)+\ldots\end{aligned}$$

Putting herein $l = 1, 2, \ldots$ in succession, we establish that

$$L_s(\infty ; a_1, a_2, \ldots, a_m) = L_s(\infty ; b_1, b_2, \ldots, b_n),$$

and thence

$$L_s(l ; a_1, a_2, \ldots, a_m) = L_s(l ; b_1, b_2, \ldots, b_n),$$

proving that the sub-lattice functions also do not change in passing from a lattice to the conjugate lattice.

As a rule, with some exceptions, the inner lattice function changes in passing from a lattice to its conjugate.

Thus it will be found that

$$\mathrm{IL}(\infty\,;\,22221) = \frac{(5)}{(2)}\,\mathrm{IL}(\infty\,;\,54),$$

(22221) and (54) being conjugate partitions.

Exceptionally, if it be proved that the inner lattice function of a complete lattice is unity, the function obviously does not change on passing to the conjugate lattice.

Another exception *appears* to be

$$\mathrm{IL}(\infty\,;\,m1^n) = \mathrm{IL}(\infty\,;\,n+1\,.\,1^{n-1}),$$

$m1^n$ and $n+1\,.\,1^{m-1}$ being conjugate partitions and there may be others.

Art. 16. My next object is to obtain the lattice function for $l = \infty$ which appertains to a lattice of two unequal rows and to find the form of the inner lattice function.

The first step is to establish the relation

$$\mathrm{L}(\infty\,;\,ab) = \mathrm{L}(\infty\,;\,a,\,b-1)+x^{a+b-1}\mathrm{L}(\infty\,;\,a-1,\,b-1) \\ +x^{a+b-2}\mathrm{L}(\infty\,;\,a-2,\,b-1)+\ldots+x^{2b}\mathrm{L}(\infty\,;\,b,\,b-1).$$

Consider the Greek letter succession $\alpha^a\beta^b$, where $a \geq b$.

The whole of the permutations derived from the lattice terminate in one of the following ways

$$\beta\,;\ \beta\,|\,\alpha\,;\ \beta\,|\,\alpha^2\,;\ \ldots\ \beta\,|\,\alpha^{a-b},$$

since α cannot occur more than $a-b$ times at the end of the permutation by reason of the fundamental property of a permutation. Permutations which terminate in the manner $\beta\,|\,\alpha^s$ where $s > 0$ clearly give rise to a factor x^{a+b-s} in the associated powers of x; the other factor will be due to all of the permutations of the succession $\alpha^{a-s}\beta^b$ *which terminate with* β; that is to say, the other factor will be

$$\mathrm{L}(\infty\,;\,a-s,\,b-1).$$

Hence

$$\mathrm{L}(\infty\,;\,ab) = \mathrm{L}(\infty\,;\,a,\,b-1)+\sum_{1}^{a-b} x^{a+b-s}\mathrm{L}(\infty\,;\,a-s,\,b-1),$$

as was to be shown.

Now assume the truth of the relation

$$\mathrm{L}(\infty\,;\,as) = \frac{(1)\,(2)\,\ldots\,(a+s)}{(2)\,(3)\,\ldots\,(a+1)\,.\,(1)\,(2)\,\ldots\,(s)}\cdot\frac{x^{s+1}\,(a-s)+(1)}{(1)},$$

when $s = b-1$, for all values of a. Then

$$\mathrm{L}(\infty ; ab) = \frac{(1)(2)\dots(a+b-1)}{(2)(3)\dots(a+1)\,.\,(1)(2)\dots(b-1)} \cdot \frac{x^b(a-b+1)+(1)}{(1)}$$
$$+ x^{a+b-1} \frac{(1)(2)\dots(a+b-2)}{(2)(3)\dots(a)\,.\,(1)(2)\dots(b-1)} \cdot \frac{x^b(a-b)+(1)}{(1)}$$
$$+ \dots$$
$$+ x^{2b} \frac{(1)(2)\dots(2b-1)}{(2)(3)\dots(b+1)\,.\,(1)(2)\dots(b-1)} \cdot \frac{x^b(1)+(1)}{(1)}.$$

The right-hand side has $a-b+1$ terms; assume that the sum of the last p terms may be written

$$x^{2b} \frac{(1)(2)\dots(2b+p-1)}{(1)(2)\dots(b+p)\,.\,(1)(2)\dots(b)} \cdot (p),$$

an assumption which is obviously justified when $p = 1$; then the sum of the last $p+1$ terms ($p \gtrless a-b$) is

$$x^{2b} \frac{(1)(2)\dots(2b+p-1)\,(p)}{(1)(2)\dots(b+p)\,.\,(1)(2)\dots(b)} + x^{2b+p} \frac{(1)(2)\dots(2b+p-1)\,\{x^b(p+1)+(1)\}}{(1)(2)\dots(b+p-1)\,.\,(1)(2)\dots(b-1)};$$

and this on simplification proves to be

$$x^{2b} \frac{(1)(2)\dots(2b+p)}{(1)(2)\dots(b+p+1)\,.\,(1)(2)\dots(b)} (p+1),$$

which is a justification of the assumption. Hence the right-hand side of the expression of $\mathrm{L}(\infty ; ab)$ is, leaving out the first term,

$$x^{2b} \frac{(1)(2)\dots(a+b-1)}{(1)(2)\dots(a)\,.\,(1)(2)\dots(b)} (a-b);$$

leading to

$$\mathrm{L}(\infty ; ab) = \frac{(1)(2)\dots(a+b-1)}{(2)(3)\dots(a+1)\,.\,(1)(2)\dots(b-1)} \cdot \frac{x^b(a-b+1)+(1)}{(1)}$$
$$+ x^{2b} \frac{(1)(2)\dots(a+b-1)}{(1)(2)\dots(a)\,.\,(1)(2)\dots(b)} (a-b)$$
$$= \frac{(1)(2)\dots(a+b)}{(2)(3)\dots(a+1)\,.\,(1)(2)\dots(b)} \cdot \frac{x^{b+1}(a-b)+(1)}{(1)}.$$

This result, being true when $b = 0$, is thus established universally. The outer function is of the required form, and the inner function is

$$\mathrm{IL}(\infty ; ab) = \frac{x^{b+1}(a-b)+(1)}{(1)} = 1 + x^{b+1} \frac{(a-b)}{(1)}.$$

This leads to the new result

$$\mathrm{GF}(\infty; ab) = \frac{x^{b+1}(\mathbf{a-b})+(\mathbf{1})}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{a+1})\,.\,(\mathbf{1})(\mathbf{2})\ldots(\mathbf{b})},$$

and as a particular case,

$$\mathrm{GF}(\infty; nn) = \mathrm{GF}(\infty, 2, n) = \frac{1}{(\mathbf{1})\{(\mathbf{2})(\mathbf{3})\ldots(\mathbf{n})\}^2(\mathbf{n+1})},$$

a result already known.

Art. 17. The determination of $\mathrm{L}(\infty; abc)$ presents great difficulties, so that the investigation proceeds in the path of least resistance. When the lattice is complete, the Greek letter succession is conveniently taken to be

$$\alpha_1^{\,n}\alpha_2^{\,n}\ldots\alpha_m^{\,n}.$$

It is clear that each permutation, that arises from the lattice, must terminate with α_m; hence this latter may be always deleted, and we find

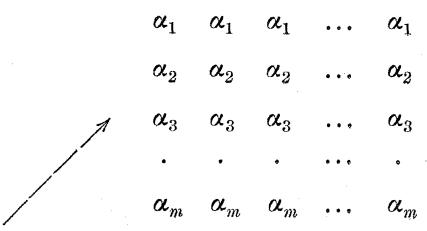
$$\mathrm{L}(\infty; n^{m-1}, n-1) = \mathrm{L}(\infty; n^m)$$

and the sub-lattice functions are also equal, but the inner lattice functions differ; thus it will be found that

$$\mathrm{IL}(\infty; nn) = 1 \qquad \text{but} \qquad \mathrm{IL}(\infty; n, n-1) = 1+x^n.$$

The Sub-lattice Functions.

Art. 18. It is necessary to inquire as to the highest order of sub-lattice function that presents itself. For a lattice of m rows and n columns I form the rectangular scheme



where there are n columns.

Reading this parallel to the arrow (inclined at 45 degrees), commencing with the left-hand top corner, I obtain the permutation

$$\alpha_1, \alpha_2 \,|\, \alpha_1, \alpha_3 \,|\, \alpha_2 \,|\, \alpha_1, \alpha_4 \,|\, \alpha_3 \,|\, \alpha_2 \,|\, \alpha_1 \ldots \alpha_m \,|\, \alpha_{m-1} \,|\, \alpha_{m-2}, \alpha_m \,|\, \alpha_{m-1}, \alpha_m.$$

This is the permutation which involves the maximum number of dividing lines and corresponds to the sub-lattice function of highest order; the permutation is unique, yielding a single power of x, which is the sub-lattice function in question. The dividing lines may be counted,

Since n^m and m^n are conjugate partitions, we may take $n \geqq m$ without loss of generality. The number of dividing lines is

$$1+2+3+\ldots+m-2+(n-m+1)(m-1)+(m-2)+\ldots+3+2+1,$$
$$=(n-1)(m-1).$$

Hence we have sub-lattice functions of all orders from zero to $(n-1)(m-1)$. It will be observed that the permutation, above written, possesses symmetry in that it is unchanged by writing α_{m-s+1} for α_s and inverting the order.

The same method is applicable to the determination of the maximum number of dividing lines appertaining to permutations derived from an incomplete lattice. Thus, if the letters be $\alpha_1^3\alpha_2^2\alpha_3$,

$$\begin{array}{lll} \alpha_1 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_2 & \\ \alpha_3 & & \end{array}$$

the reading parallel to the arrow gives

$$\alpha_1, \alpha_2 \,|\, \alpha_1, \alpha_3 \,|\, \alpha_2 \,|\, \alpha_1.$$

The highest order of sub-lattice functions when the letters are

$$\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_m^{n}$$

will be found to have higher and lower limits $\Sigma n - n_1$ and $\Sigma n - n_1 - m + 1$ respectively, the actual value depending upon the magnitudes of $n_1, n_2, \ldots, n_m$. The lower limit is the actual value when the lattice is complete.

Art. 19. The next point is the determination of the expression of $L_{(n-1)(m-1)}(\infty; n^m)$, or of $L_{(n-1)(m-1)}(\infty, m, n)$ as it may be also written.

The dividing lines occur in groups—

(i) In $m-2$ groups, containing $1, 2, \ldots, m-2$ lines respectively;
(ii) In $n-m+1$ groups, each containing $m-1$ lines;
(iii) In $m-2$ groups, containing $m-2, m-1, \ldots, 2, 1$ lines respectively.

Let the exponent of x sought be $\pi_1+\pi_2+\pi_3$; π_1, π_2, π_3, corresponding to (i), (ii), and (iii), respectively.

$$\pi_1 = 2+(4+5)+(7+8+9)+\ldots+\{\tfrac{1}{2}(m^2-3m+4)+\ldots+\tfrac{1}{2}(m^2-m-2)\},$$
$$=\tfrac{1}{2}(1\,.\,2^2+2\,.\,3^2+3\,.\,4^2+\ldots \text{ to } m-2 \text{ terms}),$$
$$=\tfrac{1}{8}(m-2)^2(m-1)^2+\tfrac{1}{6}(m-2)(m-1)(2m-3)+\tfrac{1}{4}(m-2)(m-1),$$
$$=\tfrac{1}{24}(m)(m-1)(m-2)(3m-1).$$

$$\pi_2 = \tfrac{1}{2}(m-1)m^2+\tfrac{1}{2}(m-1)m(m+2)+\tfrac{1}{2}(m-1)m(m+4)+\ldots \text{ to } n-m+1 \text{ terms},$$
$$=\tfrac{1}{2}(m-1)mn(n-m+1).$$

$$\pi_3 = (mn-2)+(mn-4+mn-5)+(mn-7+mn-8+mn-9)+\ldots \text{ to } m-2 \text{ terms},$$
$$=\tfrac{1}{2}mn(m-1)(m-2)-\tfrac{1}{24}m(m-1)(m-2)(3m-1).$$

Whence

$$\pi_1+\pi_2+\pi_3 = \tfrac{1}{2}n(n-1)m(m-1);$$

and

$$\mathrm{L}_{(n-1)(m-1)}(\infty, m, n) = x^{\frac{1}{2}n(n-1)m(m-1)}.$$

If, in the succession

$$\alpha_1, \alpha_2 | \alpha_1, \alpha_3 | \alpha_2 | \alpha_1, \ldots, \alpha_m | \alpha_{m-1}, \alpha_m,$$

we fix upon any dividing line and arrange the letters to the right of it in alphabetical order, thus obliterating the lines to the right of the one fixed upon, we obtain a permutation involving (suppose) s lines which yields x to the lowest power that occurs in the sub-lattice function of order s. When the lattice is complete we may, in any derived permutation, write α_{m-s+1} for α_s and invert the order, and we thus obtain another permutation belonging to the same sub-lattice function as the former. For a succession $\alpha_p | \alpha_q$ $p > q$ in the former becomes by the stated operations $\alpha_{m-q+1} | \alpha_{m-p+1}$, where $m-q+1 > m-p+1$ in the latter; and if α_p is the k^{th} letter from the left of the former permutation, α_{m-q+1} is the $mn-k^{\text{th}}$ letter from the left of the latter. Hence, if the power of x given by the former permutation be

$$x^{p_1+p_2+\ldots+p_s},$$

that given by the latter is

$$x^{mns-p_1-p_2-\ldots-p_s}.$$

Thus, for every term x^p in L_s, there is a corresponding term x^{mns-p}.

Hence we may say that L_s is centrically symmetrical both as regards the powers of x and the coefficients.

If e be the lowest power of x in L_s, determined as above, the highest power of x will be $mns-e$.

Ex. gr.,

$$\mathrm{L}_0(\infty, 3, 3) = 1,$$
$$\mathrm{L}_1(\infty, 3, 3) = x^2+2x^3+2x^4+2x^5+2x^6+x^7,$$
$$\mathrm{L}_2 = 2(x^6+x^7+2x^8+2x^9+2x^{10}+x^{11}+x^{12}),$$
$$\mathrm{L}_3 = x^{11}+2x^{12}+2x^{13}+2x^{14}+2x^{15}+x^{16},$$
$$\mathrm{L}_4 = x^{18};$$

and it will be noted that

$$\text{in } \mathrm{L}_1,\ x^k \text{ and } x^{9-k};\ \text{in } \mathrm{L}_2,\ x^k \text{ and } x^{18-k};\ \text{in } \mathrm{L}_3,\ x^k \text{ and } x^{27-k};$$

occur in pairs, whilst the theorem is clearly verified in L_0 and in L_4.

The result of writing $\frac{1}{x}$ for x in L_s is the acquisition of the factor x^{-mns} by L_s.

Art. 20. Let e_s and f_s be the least and greatest exponents of x that occur in the expression of $\mathrm{L}_s(\infty, m, n)$. Consider again the permutation

$$\alpha_1, \alpha_2 | \alpha_1, \alpha_3 | \alpha_2 | \alpha_1, \ldots, \alpha_m | \alpha_{m-1}, \alpha_m;$$

e_s is the exponent of x due to the dividing lines when only the first s lines from the left are retained, the letters to the right of the s^{th} line being arranged in alphabetical order. If $\mu = (m-1)(n-1)$ we know that $e_\mu = f_\mu = \frac{1}{2}mn\mu$. What is the relation between e_s and $e_{\mu-s}$? To obtain $e_{\mu-s}$ we must clearly obliterate the last s lines on the right and arrange the affected letters in alphabetical order. Since the number of letters is mn, if for e_s we retain s lines which give

$$e_s = p_1+p_2+\ldots+p_s,$$

we must for $e_{\mu-s}$ reject s lines of power values

$$mn-p_1,\ mn-p_2,\ \ldots,\ mn-p_s.$$

Hence

$$e_{\mu-s} = e_\mu - smn + e_s = e_s + \tfrac{1}{2}mn(\mu-2s);$$

and from the symmetry of the permutation we find also

$$f_{\mu-s} = f_s + \tfrac{1}{2}mn(\mu-2s);$$

so that

$$x^{e_{\mu-s}} + x^{f_{\mu-s}} = x^{\frac{1}{2}mn(\mu-2s)}\left(x^{e_s}+x^{f_s}\right);$$

an interesting result which foreshadows the theorem

$$\mathrm{L}_{\mu-s} = x^{\frac{1}{2}mn(\mu-2s)}\,\mathrm{L}_s.$$

The circumstance that the lattice function, when the lattice is complete, involves x to the power $\frac{1}{2}mn(m-1)(n-1)$, which is the greatest exponent of x that occurs in the outer function, is consistent with the inner function being simply unity.

Art. 21. I will now investigate an expression for $\mathrm{L}_1(\infty, m, n)$.

Suppose that a certain power of x arises therein from the conjunction α_v/α_u, where $m \geqq v > u$, and let the Greek letter succession be

$$\mathrm{A})\ \alpha_v \,|\, \alpha_u\ (\mathrm{B},$$

where in the space A there is any suitable succession of letters in ascending order (of subscripts) to v, and in the space B any suitable succession such that the subscripts are in ascending order from u.

The least power of x is obtained when in the space A there is the succession

$$\alpha_1{}^n\alpha_2{}^n\ldots\alpha^n{}_{u-1}\alpha_u\alpha_{u+1}\ldots\alpha_{v-1}.$$

This gives the term $x^{v+(n-1)u-n+1}$.

The greatest power arises when in the space A is

$$\alpha_1{}^n\alpha_2{}^n\ldots\alpha^n{}_{u-1}\alpha_u{}^{n-1}\alpha_{u+1}^{n-1}\ldots\alpha_{v-1}^{n-1}\alpha_v{}^{n-2};$$

and this gives the term $x^{(n-1)v+u-1}$.

It must be remembered, in assigning these successions to the space A, that only one dividing line is to be in the whole permutation, and that the latter must possess the fundamental property which is the attribute of all such.

Writing the above succession

$$\alpha_1{}^n\alpha_2{}^n\ldots\alpha^n{}_{u-1}\alpha_u\alpha_{u+1}\ldots\alpha_{v-1}\alpha_v\,|\,\alpha_u\ldots,$$
$$\alpha_1{}^n\alpha_2{}^n\ldots\alpha^n{}_{u-1}\alpha_u{}^{n-1}\alpha_{u+1}^{n-1}\ldots\alpha_{v-1}^{n-1}\alpha_v{}^{n-1}\,|\,\alpha_u\ldots,$$

we have to determine all of the successions ranging from

$$\alpha_u\alpha_{u+1}\ldots\alpha_{v-1}\alpha_v \quad \text{to} \quad \alpha_u{}^{n-1}\alpha_{u+1}^{n-1}\ldots\alpha_{v-1}^{n-1}\alpha_v{}^{n-1}$$

that may be placed between $\alpha_1{}^n\alpha_2{}^n\ldots\alpha^n{}_{u-1}$ and the dividing line in order to form an L_1 succession involving $\alpha_v\,|\,\alpha_u$.

Since $u, u+1, \ldots, v$ is a succession of $v-u+1$ numbers in ascending order we are clearly concerned with the whole of the partitions at the points of a one-row lattice of $v-u+1$ nodes which are such that the part magnitude lies between $n-2$ and zero.

If $g_1g_2\ldots g_{v-u+1}$ be one such partition

$$\Sigma x^{\Sigma g} = \frac{(n-1)\,(n)\ldots(n+v-u-1)}{(1)\,(2)\ldots(v-u+1)}.$$

Denote by $L_{1,vu}$ that portion of L_1 which is associated with the conjunction $\alpha_v\,|\,\alpha_u$; then

$$L_{1,vu} = x^{v+(n-1)u-n+1}\,\frac{(n-1)\,(n)\ldots(n+v-u-1)}{(1)\,(2)\ldots(v-u+1)};$$

leading to

$$L_1(\infty, m, n) = \sum_v\sum_u x^{v+(n-1)u-n+1}\,\frac{(n-1)\,(n)\ldots(n+v-u-1)}{(1)\,(2)\ldots(v-u+1)}.$$

Put herein $v = u+j$, then, for a constant value of j,

$$L_1(\infty, m, n)_j = \{x^{j+1}+x^{j+n+1}+\ldots+x^{nm-(n-1)(j+1)}\}\,\frac{(n-1)\,(n)\ldots(n+j-1)}{(1)\,(2)\ldots(j+1)}$$
$$= x^{j+1}\,\frac{(nm-nj)}{(n)}\cdot\frac{(n-1)\,(n)\ldots(n+j-1)}{(1)\,(2)\ldots(j+1)};$$

and, consequently, giving j all values from 1 to $m-1$,

$$L_1(\infty, m, n) = x^2\,\frac{(nm-n)}{(n)}\cdot\frac{(n-1)\,(n)}{(1)\,(2)}$$
$$+x^3\,\frac{(nm-2n)}{(n)}\cdot\frac{(n-1)\,(n)\,(n+1)}{(1)\,(2)\,(3)}$$
$$+\ldots$$
$$+x^m\,\frac{(n)}{(n)}\cdot\frac{(n-1)\,(n)\ldots(n+m-2)}{(1)\,(2)\ldots(m)}.$$

Art. 22. It is not, I think, immediately obvious that this series can be effectively summed. That this is, in fact, the case appears when the problem is looked at from another point of view.

I shall show that

$$L_1(\infty, m, n) = \frac{(n+1)(n+2)\dots(n+m)}{(1)(2)\dots(m)} - \frac{(mn+1)}{(1)};$$

for suppose that the permutation

$$\alpha_1^{k_1}\alpha_2^{k_2}\dots\alpha_m^{k_m} \mid \alpha_1^{n-k_1}\alpha_2^{n-k_2}\dots\alpha_m^{n-k_m}$$

gives rise to the term $x^{\Sigma k}$, where $(k_1k_2\dots k_m)$ is a one-row partition of Σk. The part to the left of the dividing line may be as small as $\alpha_1\alpha_2$, and the part to the right as small as $\alpha_{m-1}\alpha_m$; hence Σk has values ranging from 2 to $mn-2$; the partition $(k_1k_2\dots k_m)$ has parts limited in magnitude to n and in number to m; and if there were no deductions from the total of partitions the value of $L_1(\infty, m, n)$ would be

$$\frac{(n+1)(n+2)\dots(n+m)}{(1)(2)\dots(m)};$$

but there must be deductions, because every partition of the form n^ij must be absent; for the corresponding succession of letters is

$$\alpha_1^n\alpha_2^n\dots\alpha_i^n\alpha_{i+1}^j;$$

and a dividing line cannot be placed after this succession because every letter prior to α_{i+1} has *already* appeared to the left of α_{i+1}^j. Of these omitted partitions there is one, and only one, of each weight, viz.:—

For

$i = 0$ $\quad 1, \quad \alpha_1, \quad \alpha_1^2, \dots \alpha_1^n,$

$i = 1$ $\quad \alpha_1^n\alpha_2, \quad \alpha_1^n\alpha_2^2, \dots \alpha_1^n\alpha_2^n,$

$i = 2$ $\quad \alpha_1^n\alpha_2^n\alpha_3, \quad \alpha_1^n\alpha_2^n\alpha_3^2, \dots \alpha_1^n\alpha_2^n\alpha_3^n,$

$i = m-1$ $\quad \alpha_1^n\alpha_2^n\dots\alpha_{m-1}^n\alpha_m, \dots \alpha_1^n\alpha_2^n\dots\alpha_m^n.$

Hence we must subtract

$$\frac{(mn+1)}{(1)};$$

and

$$L_1(\infty, m, n) = \frac{(n+1)(n+2)\dots(n+m)}{(1)(2)\dots(m)} - \frac{(mn+1)}{(1)}.$$

Since $L_0 = 1$, we have also

$$L_0(\infty, m, n) + L_1(\infty, m, n) = \frac{(n+1)(n+2)\dots(n+m)}{(1)(2)\dots(m)} - x\frac{(mn)}{(1)}.$$

The Fundamental Relation.

Art. 23. We must now examine the fundamental relation

$$\mathrm{GF}(l, m, n) = \frac{(\mathrm{l}+1)(\mathrm{l}+2)\ldots(\mathrm{l}+\mathrm{mn})\,\mathrm{L}_0 + (\mathrm{l})(\mathrm{l}+1)\ldots(\mathrm{l}+\mathrm{mn}-1)\,\mathrm{L}_1 + \ldots}{\ldots + (\mathrm{l}-\mu+1)(\mathrm{l}-\mu+2)\ldots(\mathrm{l}-\mu+\mathrm{mn})\,\mathrm{L}_\mu}{(1)(2)\ldots(\mathrm{mn})}$$

$$= \frac{l_0\mathrm{L}_0 + l_1\mathrm{L}_1 + \ldots + l_\mu\mathrm{L}_\mu}{(1)(2)\ldots(\mathrm{mn})}$$

for brevity, where

$$l_s = (\mathrm{l}-\mathrm{s}+1)(\mathrm{l}-\mathrm{s}+2)\ldots(\mathrm{l}-\mathrm{s}+\mathrm{mn})$$

and

$$\mu = (m-1)(n-1).$$

When a partition enumerated by $\mathrm{GF}(l, m, n)$ is represented graphically by nodes in three dimensions, we see that the nodes form a portion of a parallelopiped of nodes, the sides having l, m, n nodes respectively; the unoccupied nodes graphically represent another partition of the number $lmn-w$ if the former partition be of the number w. Hence, if $\mathrm{GF}(l, m, n)$ be $\mathrm{F}(x)$, we have (writing Co as short for coefficient), Co x^w in $\mathrm{F}(x)$ equal to Co x^{lmn-w} in $\mathrm{F}(x)$ or equal to Co x^w in $x^{lmn}\mathrm{F}\left(\frac{1}{x}\right)$. From which it appears that $\mathrm{F}\left(\frac{1}{x}\right) = x^{-lmn}\mathrm{F}(x)$, and this property may be directly verified in the fundamental relation by means of the formulæ

$$(-\mathrm{s}) = -x^{-s}(\mathrm{s}); \quad \mathrm{L}_s\left(\frac{1}{x}\right) = x^{-smn}\mathrm{L}_s(x).$$

From another point of view we may suppose the nodes of the lattice of m rows and n columns to be all occupied by parts, zero being taken as a part, and then if we diminish each part by l, we obtain a partition of the negative integer $-(lmn-w)$ into negative parts $0, -1, -2, \ldots -l$; the effect upon the generating function $\mathrm{F}(x)$ is alternatively to substitute $\frac{1}{x}$ for x or to divide it by x^{lmn}. It will be noted that in this respect $\mathrm{L}_s(\infty, m, n)$ possesses the same property as $\mathrm{GF}(s, m, n)$.

The numerator function $l_0\mathrm{L}_0 + l_1\mathrm{L}_1 + \ldots + l_\mu\mathrm{L}_\mu$ has the factor

$$(\mathrm{l}+1)(\mathrm{l}+2)\ldots(\mathrm{l}+\mathrm{m}+\mathrm{n}-1)$$

which stands as a determined factor of the generating function.

Writing

$$l_s = (\mathrm{l}+1)(\mathrm{l}+2)\ldots(\mathrm{l}+\mathrm{m}+\mathrm{n}-1)\,l_s',$$

$$l_s' = (\mathrm{l}-\mathrm{s}+1)(\mathrm{l}-\mathrm{s}+2)\ldots(\mathrm{l})(\mathrm{l}+\mathrm{m}+\mathrm{n})(\mathrm{l}+\mathrm{m}+\mathrm{n}+1)\ldots(\mathrm{l}-\mathrm{s}+\mathrm{mn});$$

l_s' involving $(m-1)(n-1)$ or μ bracket factors.

I observe at this point that the substitution of $-l-m-n$ for l converts the factor $(\mathrm{l}+1)(\mathrm{l}+2)\ldots(\mathrm{l}+\mathrm{m}+\mathrm{n}-1)$ into

$$(-)^{m+n-1}x^{-(m+n-1)(l+\frac{1}{2}m+\frac{1}{2}n)}(\mathrm{l}+1)(\mathrm{l}+2)\ldots(\mathrm{l}+\mathrm{m}+\mathrm{n}-1),$$

so that it is unchanged except as to sign and a power of x *près*.

Art. 24. Some particular cases of the fundamental relation are instructive.

Thus

$$\mathrm{GF}(l,2,2)=\frac{(\mathrm{l}+1)\ldots(\mathrm{l}+4)+(\mathrm{l})\ldots(\mathrm{l}+3)\,x^2}{(1)\ldots(4)},$$

$$=\frac{(\mathrm{l}+1)(\mathrm{l}+2)^2(\mathrm{l}+3)}{(1)(2)^2(3)}=|\mathrm{LM}|_1|\mathrm{LM}|_2 \quad \text{when} \quad m=2.$$

Observe that

$$(\mathrm{l}+1)\ldots(\mathrm{l}+4)+(\mathrm{l})\ldots(\mathrm{l}+3)\,x^2=(\mathrm{l}+1)(\mathrm{l}+2)(\mathrm{l}+3)\{(\mathrm{l}+4)+(\mathrm{l})\,x^2\},$$
$$=(\mathrm{l}+1)(\mathrm{l}+2)^2(\mathrm{l}+3)\,\mathrm{L}(\infty,2,2),$$

showing that $\mathrm{L}(\infty,2,2)$ is a factor of the numerator.

It appears that in general $\mathrm{L}(\infty,m,n)$ is a factor of the numerator.

Thus

$$\mathrm{GF}(l,3,3)=\frac{(\mathrm{l}+1)(\mathrm{l}+2)^2(\mathrm{l}+3)^3(\mathrm{l}+4)^2(\mathrm{l}+5)\,\mathrm{L}(\infty,3,3)}{(1)(2)\ldots(9)},$$

and since

$$\mathrm{L}(\infty,3,3)=\frac{(6)(7)(8)(9)}{(2)(3)^2(4)}$$

$$\mathrm{GF}(l,3,3)=\frac{(\mathrm{l}+1)(\mathrm{l}+2)^2(\mathrm{l}+3)^3(\mathrm{l}+4)^2(\mathrm{l}+5)}{(1)(2)^2(3)^3(4)^2(5)}$$
$$=|\mathrm{LM}|_1|\mathrm{LM}|_2|\mathrm{LM}|_3 \quad \text{when} \quad m=3.$$

Art. 25. I have arrived at the expression for $\mathrm{GF}(l,2,n)$ in the following manner. We have

$$\mathrm{GF}(l,2,n)=\sum_{s=0}^{s=n-1}\frac{(\mathrm{l}-\mathrm{s}+1)\ldots(\mathrm{l}-\mathrm{s}+2\mathrm{n})}{(1)\ldots(2\mathrm{n})}\,\mathrm{L}_s(\infty,2,n),$$

and I determine $\mathrm{L}_s(\infty,2,n)$ from the Greek letter succession; for suppose $s=3$ and a succession to be

$$\alpha^{p_1+1}\beta^{q_1+1}|\,\alpha^{p_2+1}\beta^{q_2+1}|\,\alpha^{p_3+1}\beta^{q_3+1}|\,\alpha^{p_4+1}\beta^{q_4+1},$$

where p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4 may have all integer (including zero) values subject to the conditions

$$\Sigma p=\Sigma q=n-4;$$

$$p_1\geqq q_1;\ p_1+p_2\geqq q_1+q_2;\ p_1+p_2+p_3\geqq q_1+q_2+q_3.$$

A proper permutation with three dividing lines is thus secured, and we have to perform the summation

$$\Sigma x^{3(p_1+q_1)+2(p_2+q_2)+p_3+q_3+12},$$

which is the expression of $\mathrm{L}_3(\infty,2,n)$.

But I have shown in Part II. of this Memoir that

$$\Sigma x^{3(p_1+q_1)+2(p_2+q_2)+p_3+q_3}$$

subject to the stated conditions is GF $(n-4, 2, 3)$.

In general I thus establish that

$$\mathrm{L}_s(\infty, 2, n) = x^{s(s+1)}\mathrm{GF}(n-s-1, 2, s);$$

so that I am able to write

$$\mathrm{GF}(l, 2, n) = \sum_{s=0}^{s=n-1} x^{s(s+1)} \frac{(l-s+1)\ldots(l-s+2n)}{(1)\ldots(2n)} \mathrm{GF}(n-s-1, 2, s);$$

the expression of GF $(l, 2, n)$ is thus made to depend upon the expression of GF $(l', 2, n')$, where l', n' have all values such that

$$l'+n' = n-1.$$

Now assume

$$\mathrm{GF}(l', 2, n') = \frac{(l'+1)\{(l'+2)\ldots(l'+n')\}^2(l'+n'+1)}{(1)\{(2)\ldots(n')\}^2(n'+1)},$$

so that

$$\mathrm{GF}(l, 2, n) = \sum_{0}^{n-1} x^{s(s+1)} \frac{(l-s+1)\ldots(l-s+2n)}{(1)\ldots(2n)} \cdot \frac{(n-s)\{(n-s+1)\ldots(n-1)\}^2(n)}{(1)\{(2)\ldots(s)\}^2(s+1)},$$

or

$$\begin{aligned}\frac{(1)\ldots(2n)}{(l+1)\ldots(l+n+1)}\mathrm{GF}(l, 2, n) = {} & (l+n+2)\ldots(l+2n) \\ & + (l)(l+n+2)\ldots(l+2n-1)x^2\frac{(n-1)(n)}{(1)(2)} \\ & + (l-1)(l)(l+n+2)\ldots(l+2n-2)x^6\frac{(n-2)(n-1)^2(n)}{(1)(2)^2(3)} \\ & + \ldots \\ & + (l-n+2)\ldots(l)x^{(n-1)n}.\end{aligned}$$

It is easy to show that $(l+2)$, $(l+3)$, ... $(l+n)$ are each of them factors of the right-hand side; the remaining factor is free from l, and is, in fact, the lattice function L $(\infty, 2, n)$.

Giving l to the special value zero, we readily find

$$\mathrm{L}(\infty, 2, n) = \frac{(n+2)\ldots(2n)}{(2)\ldots(n)},$$

and thence

$$\mathrm{GF}(l, 2, n) = \frac{(l+1)\{(l+2)\ldots(l+n)\}^2(l+n+1)}{(1)\{(2)\ldots(n)\}^2(n+1)},$$

as was to be shown.

This proof rests upon the assumption that the law can be shown to hold for GF $(n-s-1, 2, s)$ for all values of s from 0 to $n-1$, and for all values of n. Now suppose the law to hold for GF $(l, 2, \nu)$ for all values of ν inferior to n; then it obviously holds for GF $(n-s-1, 2, s)$ for all values of s inferior to n, and thence, as has just been proved, it holds for GF $(l, 2, n)$; but the law does hold when $\nu = 0$ or 1, and thence by induction the law holds in general. This method of proof seems to be of application only when $m = 2$, for then only can the function $\mathrm{L}_s(\infty, 2, n)$ be identified with a form GF $(l', 2, n')$ where the sum of $l'+n'$ is less than n.

Art. 26. I turn again to the relation

$$\mathrm{GF}(l, m, n) = \frac{l_0\mathrm{L}_0 + l_1\mathrm{L}_1 + \ldots + l_\mu\mathrm{L}_\mu}{(1)(2)\ldots(mn)}$$

in order to establish relations between the functions GF (l, m, n) and the sub-lattice functions $\mathrm{L}_s(\infty, m, n)$. The relation, as it stands, exhibits GF (l, m, n) as a linear function of the sub-lattice functions, but giving l the special values $0, 1, 2, \ldots$ in succession, we obtain

$$\mathrm{GF}(0, m, n) = \mathrm{L}_0(\infty, m, n) = 1,$$

$$\mathrm{GF}(1, m, n) = \frac{(mn+1)}{(1)} + \mathrm{L}_1(\infty, m, n),$$

$$\mathrm{GF}(2, m, n) = \frac{(mn+1)(mn+2)}{(1)(2)} + \frac{(mn+1)}{(1)}\mathrm{L}_1 + \mathrm{L}_2,$$

. . . . . .

$$\mathrm{GF}(\mu, m, n) = \frac{(mn+1)\ldots(mn+\mu)}{(1)\ldots(\mu)} + \frac{(mn+1)\ldots(mn+\mu-1)}{(1)\ldots(\mu-1)}\mathrm{L}_1 + \ldots\ldots + \mathrm{L}_\mu,$$

and thence

$$\mathrm{L}_0 = 1,$$

$$\mathrm{L}_1 = \mathrm{GF}(1, m, n) - \frac{(mn+1)}{(1)},$$

$$\mathrm{L}_2 = \mathrm{GF}(2, m, n) - \frac{(mn+1)}{(1)}\mathrm{GF}(1, m, n) + x\frac{(mn)(mn+1)}{(1)(2)},$$

. . . . . .

$$\mathrm{L}_\mu = \mathrm{GF}(\mu, m, n) - \frac{(mn+1)}{(1)}\mathrm{GF}(\mu-1, m, n) + x\frac{(mn)(mn+1)}{(1)(2)}\mathrm{GF}(\mu-2, m, n),$$

$$- \ldots\ldots + (-)^k x^{\frac{1}{2}(k-1)k}\frac{(mn-k+2)\ldots(mn+1)}{(1)\ldots(k)}\mathrm{GF}(\mu-k, m, n)$$

$$+ \ldots$$

$$+ (-)^\mu x^{\frac{1}{2}(\mu-1)\mu}\frac{(mn-\mu+2)\ldots(mn+1)}{(1)\ldots(\mu)}.$$

Since $L_s = 0$, when $s > \mu$, the series may be continued,

$$0 = GF(\mu+1, m, n) - \frac{(mn+1)}{(1)} GF(\mu, m, n) + \ldots + (-)^{\mu+1} x^{\frac{1}{2}\mu(\mu+1)} \frac{(mn-\mu+1)\ldots(mn+1)}{(1)\ldots(\mu+1)};$$

and for values of l ranging from $\mu+1$ to mn,

$$0 = GF(l, m, n) - \frac{(mn+1)}{(1)} GF(l-1, m, n) + \ldots + (-)^{l} x^{\frac{1}{2}(l-1)l} \frac{(mn-l+1)\ldots(mn+1)}{(1)\ldots(1+1)},$$

the series having $l+1$ terms; but, when $l > mn$,

$$0 = GF(l, m, n) - \frac{(mn+1)}{(1)} GF(l-1, m, n) + \ldots + (-)^{mn+1} x^{\frac{1}{2}mn(mn+1)} GF(l-mn-1, m, n),$$

the series having $mn+2$ terms.

Art. 27. We have thus a number of difference equations satisfied by the functions $GF(l, m, n)$, and we can now show that if

$$GF(l, m, n) = |LM|_1 |LM|_2 \ldots |LM|_n = J(l, m, n)$$

for all values of l not exceeding mn, the law is true universally.

For $J(l, m, n)$ is of the form

$$P_0 - P_1 x^l + P_2 x^{2l} - \ldots (-)^{mn} P_{mn} x^{lmn},$$

where the coefficients P are functions of x independent of l.

Then

$$\sum_0^\infty J(l, m, n)\, \theta^l = \frac{P_0}{1-\theta} - \frac{P_1}{1-\theta x} + \frac{P_2}{1-\theta x^2} - \ldots\ldots (-)^{mn} \frac{P_{mn}}{1-\theta x^{mn}};$$

from which it appears that

$$(1-\theta)(1-\theta x)\ldots(1-\theta x^{mn}) \sum_0^\infty J(l, m, n)\, \theta^l$$

is of degree mn in θ, at most, and hence, when $l > mn$, since

$$(1-\theta)(1-\theta x)\ldots(1-\theta x^{mn}) = 1 - \theta\frac{(mn+1)}{(1)} + \theta^2 x \frac{(mn)(mn+1)}{(1)(2)} - \ldots\ldots$$

we have

$$J(l, m, n) - \frac{(mn+1)}{(1)} J(l-1, m, n) + \ldots + (-)^{mn+1} x^{\frac{1}{2}mn(mn+1)} J(l-mn-1, m, n) = 0;$$

but it has been shown that $l > mn$

$$GF(l, m, n) - \frac{(mn+1)}{(1)} GF(l-1, m, n) + \ldots\ldots + (-)^{mn+1} x^{\frac{1}{2}mn(mn+1)} GJ(l-mn-1, m, n) = 0.$$

Assume that $\mathrm{GF}(l, m, n) = \mathrm{J}(l, m, n)$ when l does not exceed mn; then putting $l = mn+1$ in our equations, we find that

$$\mathrm{GF}(mn+1, m, n) = \mathrm{J}(mn+1, m, n);$$

and thence by induction

$$\mathrm{GF}(l, m, n) = \mathrm{J}(l, m, n) \quad \text{when} \quad l > mn.$$

Art. 28. I now write the fundamental relation

$$\mathrm{GF}(l, m, n) = \frac{l_0\mathrm{L}_0 + l_1\mathrm{L}_1 + \ldots + l_\mu\mathrm{L}_\mu}{(1)(2)\ldots(\mathbf{mn})},$$

in the form

$$\mathrm{GF}(l, m, n) = \frac{\mathrm{E}_l}{\mathrm{E}_0},$$

where

$$\mathrm{E}_l = l_0\mathrm{L}_0 + l_1\mathrm{L}_1 + \ldots + l_\mu\mathrm{L}_\mu,$$

and assume that $\mathrm{GF}(l, m, n)$ is a product of powers of factors of the two types $1-x^{l+s}$, $1-x^s$, or $(\mathrm{l}+\mathbf{s})$, $(\mathbf{s})$, where the powers may be positive or negative integers.

I thus write

$$\frac{\mathrm{E}_l}{\mathrm{E}_0} = \Pi_1(\mathrm{l}+\mathbf{s}) \,.\, \Pi_2(\mathbf{s}),$$

leading to

$$\frac{\mathrm{E}_{l+1}}{\mathrm{E}_l} = \frac{\Pi_1(\mathrm{l}+\mathbf{s}+1)}{\Pi_1(\mathrm{l}+\mathbf{s})}.$$

Now

$$\frac{\mathrm{E}_1}{\mathrm{E}_0} = \frac{(\mathbf{mn}+1)}{(1)} + \mathrm{L}_1 = \frac{(\mathbf{n}+1)(\mathbf{n}+2)\ldots(\mathbf{n}+\mathbf{m})}{(1)(2)\ldots(\mathbf{m})},$$

therefore

$$\frac{\Pi_1(\mathbf{s}+1)}{\Pi_1(\mathbf{s})} = \frac{(\mathbf{n}+1)(\mathbf{n}+2)\ldots(\mathbf{n}+\mathbf{m})}{(1)(2)\ldots(\mathbf{m})}$$

and

$$\frac{\mathrm{E}_{l+1}}{\mathrm{E}_l} = \frac{(\mathrm{l}+\mathbf{n}+1)(\mathrm{l}+\mathbf{n}+2)\ldots(\mathrm{l}+\mathbf{n}+\mathbf{m})}{(\mathrm{l}+1)(\mathrm{l}+2)\ldots(\mathrm{l}+\mathbf{m})} = |\mathrm{NM}|_{l+1};$$

therefore

$$\frac{\mathrm{E}_{l+2}}{\mathrm{E}_l} = |\mathrm{NM}|_{l+1}|\mathrm{NM}|_{l+2},$$

and

$$\frac{\mathrm{E}_{l+l'}}{\mathrm{E}_l} = |\mathrm{NM}|_{l+1}|\mathrm{NM}|_{l+2}\ldots|\mathrm{NM}|_{l+l'}.$$

Putting herein $l = 0$, $l' = l$,

$$\frac{\mathrm{E}_l}{\mathrm{E}_0} = |\mathrm{NM}|_1|\mathrm{NM}|_2\ldots|\mathrm{NM}|_l.$$

Hence

$$\mathrm{GF}(l, m, n) = |\mathrm{NM}|_1|\mathrm{NM}|_2\ldots|\mathrm{NM}|_l,$$

and on the assumption as to form the main theorem is established.

Art. 29. We may now obtain properties of the sub-lattice functions—

For, in the relation

$$\frac{\mathrm{E}_l}{\mathrm{E}_0} = |\mathrm{NM}|_1 |\mathrm{NM}|_2 \ldots |\mathrm{NM}|_l,$$

put $l = -l-m-n$ and observe that $(-\mathbf{s}) = -x^{-s}(\mathbf{s})$.

We find

$$\mathrm{E}_{-l-m-n} = \frac{(-)^{mn}}{x^{lmn+\frac{1}{2}mn(mn+1)}} \{(\mathbf{l}+1)\ldots(\mathbf{l}+\mathbf{mn})\,\mathrm{L}_\mu + x^{mn}(\mathbf{l})\ldots(\mathbf{l}+\mathbf{mn}-1)\,\mathrm{L}_{\mu-1} + \ldots + x^{\mu mn}(\mathbf{l}-\mu+1)\ldots(\mathbf{l}+\mathbf{m}+\mathbf{n}-1)\,\mathrm{L}_0\};$$

and $|\mathrm{NM}|_1 |\mathrm{NM}|_2 \ldots |\mathrm{NM}|_l$ is unaltered to a factor $\frac{(-)^{mn}}{x^{lmn+\frac{1}{2}mn(m+n)}}$ *près.*

Hence the identity

$$(\mathbf{l}+1)\ldots(\mathbf{l}+\mathbf{mn})\,\mathrm{L}_\mu + x^{mn}(\mathbf{l})\ldots(\mathbf{l}+\mathbf{mn}-1)\,\mathrm{L}_{\mu-1} + \ldots + x^{\mu mn}(\mathbf{l}-\mu+1)\ldots(\mathbf{l}-\mu+\mathbf{mn})\,\mathrm{L}_0 = x^{\frac{1}{2}\mu mn}\{(\mathbf{l}+1)\ldots(\mathbf{l}+\mathbf{mn})\,\mathrm{L}_0 + (\mathbf{l})\ldots(\mathbf{l}+\mathbf{mn}-1)\,\mathrm{L}_1 + \ldots + (\mathbf{l}-\mu+1)\ldots(\mathbf{l}-\mu+\mathbf{mn})\,\mathrm{L}_\mu\}.$$

Herein putting

$$\begin{aligned}
l = 0 \quad &\text{we find} \quad \mathrm{L}_\mu = x^{\frac{1}{2}\mu mn}\,\mathrm{L}_0,\\
l = 1 \quad &,, \qquad \mathrm{L}_{\mu-1} = x^{\frac{1}{2}(\mu-2)mn}\,\mathrm{L}_1,\\
l = 2 \quad &,, \qquad \mathrm{L}_{\mu-2} = x^{\frac{1}{2}(\mu-n)mn}\,\mathrm{L}_2,\\
&\ldots\ldots\ldots\ldots\\
l = t \quad &,, \qquad \mathrm{L}_{\mu-t} = x^{\frac{1}{2}(\mu-2t)mn}\,\mathrm{L}_t,
\end{aligned}$$

exhibiting an elegant property of the functions.

From the relation

$$\frac{\mathrm{E}_l}{\mathrm{E}_0} = |\mathrm{NM}|_1 |\mathrm{NM}|_2 \ldots |\mathrm{NM}|_l$$

we find, giving l the values 0, 1, 2, ... in succession,

$$\begin{aligned}
\mathrm{L}_0 &= 1,\\
\mathrm{L}_1 &= |\mathrm{NM}|_1 - \frac{(\mathbf{mn}+1)}{(\mathbf{1})},\\
\mathrm{L}_2 &= |\mathrm{NM}|_1 |\mathrm{NM}|_2 - \frac{(\mathbf{mn}+1)}{(\mathbf{1})}|\mathrm{NM}|_1 + x\frac{(\mathbf{mn})(\mathbf{mn}+1)}{(\mathbf{1})(\mathbf{2})},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\mathrm{L}_\mu &= |\mathrm{NM}|_1 |\mathrm{NM}|_2 \ldots |\mathrm{NM}|_\mu - \frac{(\mathbf{mn}+1)}{(\mathbf{1})}|\mathrm{NM}|_1 |\mathrm{NM}|_2 \ldots |\mathrm{NM}|_{\mu-1}\\
&\quad + \ldots + (-)^\mu x^{\frac{1}{2}(\mu-1)\mu}\frac{(\mathbf{mn}-\mu+2)\ldots(\mathbf{mn}+1)}{(\mathbf{1})\ldots(\mu)}
\end{aligned}$$

The relations between the functions L_s and $L_{\mu-s}$ yield remarkable algebraical identities.

So far I have established a new method for the discussion of these questions of the arrangements of numbers, and have made some progress with the simplification of the fundamental expression arrived at for the generating function. I have further shown the great probability of the outer lattice function being the whole lattice function whenever the lattice is complete. Before this can be rigidly established, I believe that a further study of the theory of the incomplete lattice will be necessary. From many particular incomplete lattices that have already been worked out this investigation promises well, and I hope in due course to lay the results before the Society.

Postscript.

There is an analogous theory which is concerned with the totality of the permutations of $\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_n^{p_n}$. We thus obtain permutation functions which possess elegant properties. The functions also arise from the theory of partitions.

Suppose that we desire the number of two-dimensional partitions of a number such that the nodes of the lattice descending order of part magnitude is in evidence in each row but *not necessarily* in each column. It is immediately evident that the generating function of such at the nodes of a lattice which contains $p_1, p_2, \ldots, p_n$ nodes in the successive rows is

$$\frac{1}{(1)\ldots(p_1)\,(1)\ldots(p_2)\ldots\ldots(1)\ldots(p_n)},$$

whether the numbers $p_1, p_2, \ldots, p_n$ be in descending order of magnitude or not. This fact enables us to determine the lattice function and the sub-lattice functions derivable from the whole of the permutations of the letters $\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_n^{p_n}$ when we may suppose the exponents $p_1, p_2, \ldots, p_n$ to be in descending order of magnitude and establishes also that *these functions are invariant for any permutation of* $p_1, p_2, \ldots, p_n$ *in the product* $\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_n^{p_n}$.

We may proceed in exactly the same manner as when the restricted permutations were under view. Taking the lattice corresponding to $\alpha_1^3\alpha_2^2\alpha_3$ and arranging 6 different numbers in any way so that descending order is in evidence in the rows

$$\begin{array}{lll} 3 & 2 & 1 \\ 6 & 5 & \\ 4 & & \end{array} \qquad \alpha_2\alpha_2\alpha_3 \,|\, \alpha_1\alpha_1\alpha_1$$

we have the arrangement figured and the corresponding Greek-letter succession, yielding a portion

$$\frac{x^3}{(1)\ldots(6)}$$

of the generating function.

For the whole of the permutations derived as above from the lattice which are, in fact, the whole of the permutations of $\alpha_1^3\alpha_2^2\alpha_3$ we derive a permutation function

$$\mathrm{PF}\,(\infty; 321),$$

such that the generating function sought is

$$\frac{\mathrm{PF}\,(\infty; 321)}{(1)\ldots(6)},$$

and this we know otherwise to have the value

$$\frac{1}{(1)(2)(3).(1)(2).(1)}.$$

Hence

$$\mathrm{PF}\,(\infty; 321) = \frac{(1)(2)(3)(4)(5)(6)}{(1)(2)(3).(1)(2).(1)},$$

and, in general,

$$\mathrm{PF}\,(\infty; p_1p_2\ldots p_n) = \frac{(1)\ldots(\Sigma\mathrm{p})}{(1)\ldots(\mathrm{p}_1).(1)\ldots(\mathrm{p}_2)\ldots\ldots(1)\ldots(\mathrm{p}_n)},$$

an expression which is to be compared with the number which enumerates the permutations of the letters in $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$. The former becomes equal to the latter when $x = 1$.

When the part magnitude is limited by the number l, the enumerating generating function of the partition is

$$\frac{(l+1)\ldots(l+\mathrm{p}_1).(l+1)\ldots(l+\mathrm{p}_2)\ldots\ldots(l+1)\ldots(l+\mathrm{p}_n)}{(1)\ldots(\mathrm{p}_1).(1)\ldots(\mathrm{p}_2)\ldots\ldots(1)\ldots(\mathrm{p}_n)},$$

but, from previous work, if $\mathrm{PF}_s\,(\infty; p_1p_2\ldots p_n)$ is the sub-permutation function derived from the permutations possessing s dividing lines, this generating function is also

$$\frac{(l+1)\ldots(l+\Sigma\mathrm{p})\,\mathrm{PF}_0+(l)\ldots(l+\Sigma\mathrm{p}-1)\,\mathrm{PF}_1+\ldots\ldots+(l-\nu+1)\ldots(l-\nu+\Sigma\mathrm{p})\,\mathrm{PF}_\nu}{(1)\ldots(\Sigma p)},$$

where $\nu = \Sigma p - p_1$ (see 'Phil. Trans. Roy. Soc.,' A, vol. 207, p. 119).

Equating the two expressions for the generating function, and giving l the values 0, 1, 2, ... in succession, we find the relations

$$1 = \mathrm{PF}_0,$$

$$\frac{(\mathrm{p}_1+1)(\mathrm{p}_2+1)\ldots(\mathrm{p}_n+1)}{(1)^n} = \frac{(\Sigma\mathrm{p}+1)}{(1)} + \mathrm{PF}_1,$$

$$\frac{(\mathrm{p}_1+1)(\mathrm{p}_1+2).(\mathrm{p}_2+1)(\mathrm{p}_2+2)\ldots(\mathrm{p}_n+1)(\mathrm{p}_n+2)}{(1)^n(2)^n} = \frac{(\Sigma\mathrm{p}+1)(\Sigma\mathrm{p}+2)}{(1)(2)} + \frac{(\Sigma\mathrm{p}+1)}{(1)}\mathrm{PF}_1+\mathrm{PF}_2,$$

$$\&\mathrm{c.} = \&\mathrm{c.},$$

from which the general expression for PF_s is readily obtainable.

Putting $p_1 = p_2 = \ldots = p_n = p$ for a complete lattice, we find

$$\mathrm{PF}_0 = 1,$$

$$\mathrm{PF}_1 = \left\{\frac{(p+1)}{(1)}\right\}^n - \frac{(np+1)}{(1)},$$

$$\mathrm{PF}_2 = \left\{\frac{(p+1)(p+2)}{(1)(2)}\right\}^n - \frac{(np+1)}{(1)}\left\{\frac{(p+1)}{(1)}\right\}^n + x\frac{(np)(np+1)}{(1)(2)},$$

$$\mathrm{PF}_s = \left\{\frac{(p+1)\ldots(p+s)}{(1)\ldots(s)}\right\}^n - \frac{(np+1)}{(1)}\left\{\frac{(p+1)\ldots(p+s-1)}{(1)\ldots(s-1)}\right\}^n + \ldots + (-)^s x^{\frac{1}{2}(s-1)(s)}\frac{(np-s+2)\ldots(np+1)}{(1)\ldots(s)}.$$

A simplification, when $n = 2$, is interesting; for then

$$\mathrm{PF}_s(\infty; pp)_x = x^{s^2}\left\{\frac{(p)\ldots(p-s+1)}{(1)\ldots(s)}\right\}^2.$$

In fact, more generally it will be found that

$$\mathrm{PF}_s(\infty; pq) = x^{s^2}\frac{(p)\ldots(p-s+1).(q)\ldots(q-s+1)}{\{(1)\ldots(s)\}^2}.$$

An interesting verification is supplied by a result in a previous paper.* It was therein shown that the number of permutations of the letters composing the product

$$\alpha^p\beta^q,$$

which have s dividing lines, is the coefficient of $\lambda^s\alpha^p\beta^q$ in the expansion of the product

$$(\alpha+\lambda\beta)^p(\alpha+\beta)^q.$$

From this expression I derive a function of x, viz.,

$$(\alpha+\lambda\beta x)(\alpha+\lambda\beta x^2)\ldots(\alpha+\lambda\beta x^p).(\beta+\alpha)(\beta+\alpha x)\ldots(\beta+\alpha x^{q-1}),$$

and therein the coefficient of $\lambda^s\alpha^p\beta^q$ is readily shown to be

$$x^{s^2}\frac{(p)\ldots(p-s+1).(q)\ldots(q-s+1)}{\{(1)\ldots(s)\}^2},$$

as already obtained.

When the lattice is complete the functions PF_s possess elegant properties, just as when the permutations are restricted.

* "Memoir on the Theory of the Compositions of Numbers," 'Phil. Trans.,' A, 1893, Art. 24.

For, in the identity

$$\left\{\frac{(l+1)(l+2)\ldots(l+p)}{(1)(2)\ldots(p)}\right\}^n$$

$$= \frac{(l+1)\ldots(l+np)\,\mathrm{PF}_0 + (l)\ldots(l+np-1)\,\mathrm{PF}_1 + \ldots + (l-np+p+1)\ldots(l+p)\,\mathrm{PF}_{(n-1)p}}{(1)(2)\ldots(np)},$$

substitute $-l-p-1$ for l and we find that the left-hand side is merely multiplied by $x^{-lnp-\frac{1}{2}np(p+1)}$, whilst on the right hand the coefficient of PF_s is multiplied by $x^{-lnp+\frac{1}{2}np(np-2p-2s-1)}$. An identity thence arises, and putting therein $l_1 = 0, 1, 2, \ldots$ in succession, we find the relations

$$\mathrm{PF}_{np-p} = x^{\frac{1}{2}n(n-1)p^2}\,\mathrm{PF}_0,$$

$$\mathrm{PF}_{np-p-1} = x^{\frac{1}{2}n(n-1)p^2-np}\,\mathrm{PF}_1,$$

. . . . . .

$$\mathrm{PF}_{np-p-s} = x^{\frac{1}{2}n(n-1)p^2-snp}\,\mathrm{PF}_s,$$

giving very noteworthy algebraical identities.